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The β function in higher covariant derivative regularisation

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Abstract. We regularise a Yang-Mills theory coupled to matter by higher covariant derivatives (HCD) supplemented by dimensional regularisation. Details of its renormalisation are given. The value of the one-loop β function is computed and is found to agree with that of other regularisation schemes. We show to all orders that the β function is determined by the interactions of the original unregularised theory. The consequences that these results have for other regularisation schemes employing HCD is also discussed.

1. Introduction

In recent years, interest has been revived in higher covariant derivatives (HCD) as a possible method of consistently regularising the ultraviolet divergences of four-dimensional field theories, whilst at the same time preserving supersymmetry [1]. For a gauge theory however, such a scheme is incomplete with certain one-loop diagrams having a degree of divergence $\geqslant 0$. Therefore a secondary regulator has to be introduced to deal with these divergences and it usual to choose Pauli-Villars (PV). Indeed one can introduce the fictious Pauli-Villars fields into the theory in such a way as they only contribute (and thus regulate) one-loop diagrams. That one can do this whilst at the same time maintaining gauge invariance is not obvious but has been shown to be the case [2].

This latter scheme has been generalised to the case where it does preserve supersymmetry [3]. The resulting regularised action is written in terms of background superfields, and is rather more complicated than the non-supersymmetric case.

In this paper we study Yang-Mills theories including matter, where the regularisation scheme employs HCD and dimensional regularisation (DR) as the secondary regulator. Although dimensional regularisation is a 'complete' regulator by itself we only use it to regularise those one-loop (sub)graphs that are divergent even in the presence of HCD. Moreover calculations carried out using HCD and DR are much simpler than using Pauli-Villars as the secondary regulator.

We show how renormalisation is proved using the Ward identities of BRS transformations. Although Day [4] has analysed renormalisation for the HCD-PV scheme we adopt a somewhat different approach in dealing with potential anomalies that arise in the gauge Ward identities.

The main part of the paper then deals with the calculation of the one-loop β function, in the HCD-DR scheme. There seems to be an absence from the literature of

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explicit calculations involving HCD and two-regulator schemes in general. We therefore give a detailed interpretation of our results and in particular we prove to all orders some model-independent properties of HCD regularisation.

The paper is ordered as follows. Section 2 gives the details of renormalisation in the HCD-DR scheme. Section 3 contains the results of explicit one-loop calculations to determine the β function and the conclusions we draw from them.

2. Renormalisation in the HCD-DR scheme

Higher derivative regularisation [5] is implemented by adding to the classical action $(1/\Lambda^2)\partial^4+(1/\Lambda^4)\partial^6+\ldots$ kinetic terms for bosonic fields and $(1/\Lambda^2)\partial^2\partial+(1/\Lambda^4)\partial^4\partial+\ldots$ kinetic terms for fermionic fields. In a gauge theory, one replaces ∂_μ by the appropriate covariant derivative (and for supersymmetry, background covariant superspace derivatives [3]). This results in an improved asymptotic behaviour in the field propagators.

Apart from certain one-loop (sub)diagrams, all Feynman graphs are regulated by this procedure. To deal with the remaining (sub)divergences one introduces a second regularisation scheme.

We will consider a general Yang-Mills-fermionic matter theory that is regulated by HCD and dimensional regularisation. For the purpose of proving the renormalisability of this model one defines the action in $d=2\omega$ dimensions, so that at the outset one is dealing with an *action* that regulates the ultraviolet divergences in the theory. The action including gauge fixing and ghost terms is

$$S[\Lambda,\varepsilon]=\int \mathrm{d}^{2\omega}x\frac{\mathrm{Tr}}{C_2(G)}\Big\{-\tfrac{1}{4}(F_{\mu\nu})^2-(\gamma/4\Lambda^4)(\mathscr{D}^2F)^2$$
$$+\mathrm{i}\bar{\psi}(\gamma\mu D_\mu\psi)-\frac{1}{2\alpha}[(\partial^\mu A_\mu)^2+(\gamma/\Lambda^4)(\partial^2\partial^\mu A_\mu)^2]$$
$$-\bar{\eta}\partial^\mu(\mathscr{D}_\mu\eta)+J^\mu A_\mu+\bar{\chi}\psi+\bar{\psi}\chi+\bar{\omega}\eta+\bar{\eta}\omega\Big\} \qquad (1)$$

where in (1)

$$\mathscr{D}_\mu=\partial\mu+g[A_\mu,\ ] \qquad g=\hat{g}\mu^{2-\omega}$$
$$D_\mu\psi=(\partial\mu\psi+\mathrm{i}gA_\mu^\alpha(T^\alpha)_F\psi)$$

$\hat{g}$ being dimensionless.

$$[T_F^\alpha, T_F^\beta]=\mathrm{i}f^\alpha\beta^\gamma T_F^\gamma \qquad (\mathscr{D}^2F)_{\mu\nu}=[\mathscr{D}_\rho,[\mathscr{D}^\rho, F]]_{\mu\nu}.$$

Notice in (1) that we have not included HCD in the fermionic fields, because it turns out that the HCD in the gauge fields are sufficient to regulate all but a few diagrams with internal fermion lines and the ones that remain unregulated are not improved by the addition of fermionic derivatives [5]. Also we have not included HCD in the ghost fields but have instead adopted an equivalent procedure and introduced them into the weighting function that appears in gauge fixing. We note that supersymmetry would demand the introduction of HCD to boson and fermion fields alike; we will say more about this in § 3.

The overall degree of divergence of an arbitrary L-loop diagram can easily be computed from the interactions in (1):

$$D_{\Lambda,\omega}=6-(2\omega-6)L-\tfrac{5}{2}E_f-2E_{gh}-n_3-4n_6-5n_7-6n_8-n_f-n_{gh} \qquad (2)$$

where in (2) E_f, E_{gh} are the number of external fermion and ghost lines, n_r = number of r-point gauge vertices, n_f = number of gauge fermion vertices and η_{gh} the gauge-ghost vertices. It is clear that for $2\omega = 4$, $L \geq 2$, $D_{\Lambda,\omega} \leq 0$ so that all $L \geq 2$ loop diagrams have their *overall* divergences regularised by HCD alone whilst at $L = 1$ analyticity is maintained for sufficiently small $\omega < 2$.

The complete ultraviolet divergence of a diagram consists of an overall divergence (measured by power counting), when all of the loop momentum became hard as well as various subgraph divergences which occur when subsets of loop momentum become large, whilst the rest remain fixed. From (2) we therefore see that the only source of divergence in an $L \geq 2$ diagram comes from certain one-loop subgraphs.

We now wish to discuss the renormalisation of the theory described by (1). It will be assumed that the Weyl fermions ψ are in suitable representations of the gauge group so as to cancel the chiral anomaly. Day [4] has shown the renormalisation of (1) using HCD and PV by assuming certain properties of the theory when regularised by DR go over to the HCD-PV regularised theory. There does not seem any *a priori* reason for this and we will show here that renormalisability follows without these assumptions. Our argument holds equally for the HCD-PV scheme.

The basic tools one needs to show renormalisability are the gauge BRS ward identities along with the ghost equation of motion. The action (1) (apart from sources) is invariant under the following BRS transformations:

$$\begin{aligned}
&\delta A_\mu{}^\alpha = (\mathscr{D}_\mu \eta)^\alpha \varepsilon \\
&\delta\psi = \mathrm{i} g T_f^\alpha \eta^\alpha \psi \varepsilon \\
&\delta\bar{\psi} = -\mathrm{i} g T_f^\alpha \eta^\alpha \bar{\psi} \varepsilon \\
&\delta\eta^\alpha = -\tfrac{1}{2} g \eta^\beta \eta^\gamma f^{\alpha\beta\gamma} \varepsilon \\
&\delta\bar{\eta}^\alpha = -(1/\alpha)(1 + \gamma(\partial^2/\Lambda^2)^2)(\partial \cdot A)^\alpha \varepsilon \qquad \varepsilon^2 = 0.
\end{aligned} \tag{3}$$

Introducing the terms

$$\frac{\mathrm{Tr}}{C_2(G)} \int \mathrm{d}^{2\omega} x [K_\mu (\mathscr{D}\mu\eta) - \tfrac{1}{2} L(\eta \otimes \eta) g + \mathrm{i} g (\bar{N} \eta^\alpha T^\alpha \psi + \bar{\psi} \eta^\alpha T^\alpha M)]$$

into (1), the transformations (3) imply that the effective action $\Gamma(A_\mu, \psi, \bar{\psi}, \eta, \bar{\eta}, \alpha, \gamma, K, L, \bar{N}, M)$, which is defined such that

$$\begin{aligned}
&\frac{\delta A_\mu{}^\alpha}{\delta \varepsilon} = \frac{\delta \Gamma}{\delta K^\mu{}_\alpha} \qquad \text{etc} \ldots \\
&\frac{\delta \Gamma}{\delta A^\mu{}_\alpha} = J_\mu{}^\alpha \qquad \text{etc} \ldots
\end{aligned} \tag{4}$$

satisfies the Ward identities

$$\left(\frac{\delta\Gamma}{\delta K_\mu{}^\alpha} \frac{\delta\Gamma}{\delta A^\mu{}_\alpha} + \frac{\delta\Gamma}{\delta \bar{N}^a} \frac{\delta\Gamma}{\delta\psi^a} + \frac{\delta\Gamma}{\delta M^a} \frac{\delta\Gamma}{\delta\bar{\psi}^a} + \frac{\delta\Gamma}{\delta L^\alpha} \frac{\delta\Gamma}{\delta\eta^\alpha} \right) = \Gamma * \Gamma = 0. \tag{5}$$

From now on we think of the fields $A_\mu{}^\alpha \ldots \eta^\alpha$, as being phenomenological fields having the same transformation properties as those of the classical theory, but which minimise the full quantum action Γ. The $\bar{\eta}^\alpha$ ghost equation of motion is

$$(\partial_\mu \delta / \delta K_\mu{}^\alpha - \delta / \delta \bar{\eta}^\alpha) \Gamma = 0. \tag{6}$$

The index a runs over the dimension of the fermion representation of the gauge group G. Note that for calculational reasons the Γ appearing in (5) and (6) has the gauge fixing term removed (as is standard practice) which implies the homogeneous form of the identities (5); it can be reinstated at the end of manipulations without changing the results [6]. Therefore $\Gamma = S_0(\Lambda, \omega) + O(\hbar)$.

To proceed one expands Γ perturbatively in $\hbar$ and separates the finite and infinite parts

$$\Gamma = \sum_{n=0}^{\infty} \hbar^n \Gamma^{(n)} \qquad \Gamma^{(n)} = \Gamma^{(n)}_{\text{FINITE}} + \Gamma^{(n)}_{\text{DIV}}. \tag{7}$$

Now by gauge invariance and power counting $\Gamma^{(n)}_{\text{DIV}}$ can only have three types of infinities; $1/\varepsilon$, $(1/\varepsilon)(\ln \Lambda/\mu)^2$ and $(\ln \Lambda/\mu)^p$ where p, q are integers $\geqslant 1$ and μ is a renormalisation mass. It is clear from the linearity in Γ of the identity (6) that $\Gamma^{(n)}_{\text{DIV}}$ and $\Gamma^{(n)}_{\text{FINITE}}$ separately satisfy it. This is also true of (5) for the $(1/\varepsilon)$, $(1/\varepsilon)(\ln \Lambda/\mu)^q$ type divergences in $\Gamma^{(n)}$, but not necessarily so for the $(\ln \Lambda/\mu)^P$ type, as pointed out by Day [4]. This can be seen if we look at, say, the $O(\hbar)$ identities in (5):

$$(S_0 * \Gamma^{(1)}_{\text{DIV}} + S_0 * \Gamma^{(1)}_{\text{FINITE}} + \Gamma \leftrightarrow S_0) = 0. \tag{8}$$

Since S_0 contains $1/\Lambda^4$ factors in the HCD terms (denoted by S_0^{HCD}) these can multiply $\ln(\Lambda/\mu)$ infinities in $\Gamma^{(1)}_{\text{DIV}}$ yielding finite results that could cancel with those of the second term in (8). Hence the split (7) is ambiguous. Day argues that, in the $d=4$ case, such dangerous terms cancel because they are of the form

$$\frac{\delta S_0^{\text{HCD}}}{\delta A_\mu{}^\alpha} \frac{\delta \Gamma^{(1)}_{\text{DIV}}}{\delta K_\mu{}^\alpha} = \delta_{\text{BRS}} S_0^{HCD} = 0 \tag{9}$$

where from theories employing dimensional regularisation, $\delta\Gamma^{(n)}/\delta K_\mu{}^\alpha \propto (\mathscr{D}_\mu \eta)^\alpha$ is known to be true. However this does not seem to us to be the correct argument because there is no reason why the results (9) found using dimensional regularisation should transfer to other regularisation schemes. Moreover in the more general situation of having HCD in the fermionic terms (e.g. if we wish to preserve SUSY), the dangerous terms in the $O(\hbar)$ WI would now be

$$\frac{\delta S_0^{\text{HCD}}}{\delta A_\mu{}^\alpha} \frac{\delta \Gamma^{(1)}_{\text{DIV}}}{\delta K^\mu{}_\alpha} + \frac{\delta S_0^{\text{HCD}}}{\delta \bar{\psi}^a} \frac{\delta \Gamma^{(1)}_{\text{DIV}}}{\delta M^a} + \frac{\delta S_0^{\text{HCD}}}{\delta \psi^a} \frac{\delta \Gamma^{(1)}_{\text{DIV}}}{\delta \bar{N}^a}. \tag{10}$$

For this to vanish by gauge invariance as in (9) we would not only require

$$\frac{\delta \Gamma^{(1)}_{\text{DIV}}}{\delta A_\mu{}^\alpha} \propto (\mathscr{D}_\mu \eta)^\alpha \qquad \frac{\delta \Gamma^{(1)}_{\text{DIV}}}{\delta M^a} \propto \frac{\delta_{\text{BRS}} \bar{\psi}^a}{\delta \varepsilon} \qquad \frac{\delta \Gamma^{(1)}_{\text{DIV}}}{\delta \bar{N}^a} \propto \frac{\delta_{\text{BRS}} \psi^a}{\delta \varepsilon} \tag{11}$$

but also that they all have the same $\ln(\Lambda/\mu)$ infinity. There does not seem to be any *a priori* reason for this conspiracy to occur at one loop, quite apart from higher orders.

In fact one can argue that it is not necessary to use the action S_0 (which includes S_0^{HCD}) in order to prove renormalisability. Renormalisation based on $(S_0 - S_0^{\text{HCD}})$ is consistent if there are no $(1/\varepsilon)$ $(\ln \Lambda/\mu)^q$ or $1/\varepsilon$ infinities proportional to the HCD terms in $\Gamma^{(n)}$. Pure $\ln(\Lambda/\mu)$ type infinities are permissible, since they yield finite results in the $\Lambda \to \infty$ limit when multiplied by the overall $1/\Lambda^4$ factors occurring in S_0^{HCD}.

The reason for the absence of these infinities is exactly analogous to the absence of $\ln(\mu_p/\mu)$ and $(\ln \mu_p/\mu)(\ln \Lambda/\mu)^q$ infinities proportional to HCD terms in the HCD-PV scheme (μ_p is the Pauli–Villars mass), and so we refer the reader to [6] for details.

Let Σ_0 be the action (1) plus BRS sources but with the HCD and gauge fixing terms removed:

$$\Sigma_0 = S_0 - S_0^{\mathrm{HCD}}. \tag{12}$$

The O($\hbar$) WI becomes

$$(\Sigma_0 * \Gamma^{(1)}_{\mathrm{DIV}}) + (\Sigma_0 * \Gamma^{(1)}_{\mathrm{FINITE}}) + S_0^{\mathrm{HCD}} * \Gamma^{(1)}_{\mathrm{DIV}} = 0. \tag{13}$$

Now if $\Gamma^{(1)}_{\mathrm{DIV}}$ is pure $\ln(\Lambda/\mu)$ divergent then (13) implies

$$(\Sigma_0 * \Gamma^{(1)}_{\mathrm{DIV}}) = 0 \tag{14}$$

where ( ) in (13) and (14) indicate symmetrisation, whilst for $(1/\varepsilon)$ and mixed divergences in $\Gamma^{(1)}_{\mathrm{DIV}}$, (8) is satisfied.

We define the renormalised action $S_r^{(1)}$ in the usual way:

$$S_r^{(1)} = S_0 - \Gamma^{(1)}_{\mathrm{DIV}}. \tag{15}$$

It is clear that $S_r^{(1)}$ generates finite $\Gamma^{(1)}$ but does not satisfy the WI (5) for the reasons mentioned. However the quantity

$$\Sigma_0 - \Gamma^{(1)}_{\mathrm{DIV}} \tag{16}$$

does satisfy them up to O($\hbar^2$). We can make it satisfy (5) exactly by adding to (15) a term Q of order $\hbar$ to cancel the term $\Gamma^{(1)}_{\mathrm{DIV}} * \Gamma^{(1)}_{\mathrm{DIV}}$. To show that such a Q exists it is not necessary to directly calculate it, but to solve the identities (14) and (6) for $\Gamma^{(1)}_{\mathrm{DIV}}$. The existence of Q is then proven if (16) can be written as a renormalisation of the parameters in Σ_0, since Q will then be the O($\hbar^2$) terms in the expansion of Σ_0 in terms of the renormalised fields and coupling constants.

The inductive proof of renormalisability assumes that one has defined

$$\begin{aligned} S_r^{(n)} &\equiv S_0 - \Gamma^{(1)}_{\mathrm{DIV}} - \ldots - \Gamma^{(n)}_{\mathrm{DIV}} + \mathrm{O}(\hbar^{n+1}) \\ &= S_0(\mathring{A}_\mu, \ldots, \mathring{M}) \end{aligned} \tag{17}$$

where $S_0(A^0_\mu, \ldots)$ generates finite $\Gamma^{(1)}, \ldots, \Gamma^{(n)}$. In (17) the bare quantities $\mathring{A}^\alpha_\mu$, ψ, etc, are related to the renormalised ones through the usual relations

$$\mathring{A}_\mu{}^\alpha = Z_3{}^{1/2} A_\mu{}^\alpha$$

$$\mathring{\psi}^\alpha = Z_4{}^{1/2} \psi^a$$

etc. $Z_3, Z_4, \ldots$ contain the O($\hbar^n$) divergences. The O($\hbar^{n+1}$) WI (5) then gives

$$(\Sigma_0 * \Gamma^{(n+1)}_{\mathrm{DIV}}) + (S_0^{\mathrm{HCD}} * \tilde{\Gamma}^{(n+1)}_{\mathrm{DIV}}) = 0 \tag{18}$$

where in (18) $\tilde{\Gamma}^{(n+1)}_{\mathrm{DIV}}$ are the $1/\varepsilon$ and mixed divergences of $\Gamma^{(n+1)}$. Since as mentioned earlier $\tilde{\Gamma}^{(n+1)}_{\mathrm{DIV}}$ does not correct S_0^{HCD}, both terms in (18) must separately vanish. One has to solve for $\Gamma^{(n+1)}_{\mathrm{DIV}}$ in (13) with $\Gamma^{(1)}_{\mathrm{DIV}}$ replaced by $\Gamma^{(n+1)}_{\mathrm{DIV}}$. The most general solution is [7]

$$\begin{aligned} \Gamma^{(n+1)}_{\mathrm{DIV}} = \int \mathrm{d}^{2\omega}x \bigg(& \mathscr{L}(A_\mu, \psi, \bar{\psi}) + a(K^{\mu\alpha} - \bar{\eta}^\alpha \partial^\mu) \frac{\delta\Sigma_0}{\delta k^{\mu\alpha}} \\ & + bL^a \frac{\delta\Sigma_0}{\delta L^\alpha} + C\bar{N}^a \frac{\delta\Sigma_0}{\delta\bar{N}^a} + dM^a \frac{\delta\Sigma_0}{\delta M^a} + b\eta^\beta \frac{\delta\Sigma_0}{\delta\eta^\beta} \\ & + c\psi^a \frac{\delta\Sigma_0}{\delta\psi^a} + d\bar{\psi}^a \frac{\delta\Sigma_0}{\delta\bar{\psi}^a} + aA_\mu{}^a \frac{\delta\Sigma_0}{\delta A_\mu{}^\alpha} \bigg). \end{aligned} \tag{19}$$

In (19), $\mathscr{L}(A_\mu, \bar\psi, \psi)$ is any gauge invariant function of the fields A_μ, ψ, $\bar\psi$, having dimension $(4-2\omega)$. a, b, c, d are constants containing the various infinities in $\Gamma^{(n+1)}$. Without loss of generality one can write $\mathscr{L}(A_\mu, \bar\psi, \psi)$ by homogeneity as

$$\int d^{2\omega}x\,\mathscr{L} = \rho\bar\psi\frac{\delta\Sigma_0}{\delta\bar\psi} + \tfrac{1}{2}\sigma A_\mu{}^\alpha\frac{\delta\Sigma_0}{\delta A_\mu{}^\alpha} - \tfrac{1}{2}g\sigma\frac{\delta\Sigma_0}{\delta g}$$
$$+ (\tfrac{1}{2}\sigma - \rho)M^a\frac{\delta\Sigma_0}{\delta M^a} + \tfrac{1}{2}\sigma\bar N^a\frac{\delta\Sigma_0}{\delta N^a} + \tfrac{1}{2}\sigma L^\alpha\frac{\delta\Sigma_0}{\delta L^\alpha}. \tag{20}$$

Hence, it follows that

$$\Sigma_{0r}^{(n+1)} \equiv \Sigma_{0r}^{(n)}(\mathring{A}_\mu^0, \ldots \mathring{M}) - \Gamma_{\text{DIV}}^{(n+1)} + O(\hbar^{n+2})$$
$$= \Sigma_0(\mathring{A}'_\mu, \ldots \mathring{M}') \tag{21}$$

where in (21) the bare quantities are $\mathring{A}'_\mu = (Z'_3)^{1/2}\mathring{A}_\mu$ with $(Z'_3)^{1/2} = (Z_3 + a + \sigma/2)^{1/2}$ and so on, while $\Sigma_{0r}^{(n)}$ is $S_r^{(n)} - S_0^{\text{HCD}}$.

As it stands, the renormalised action on the LHS of (21), although containing the counterterms to $O(\hbar^{n+1})$, does not contain S_0^{HCD} and so does not regulate $\Gamma^{(n)}$.

We remedy this by adding S_0^{HCD} to $\Sigma_{0r}^{(n+1)}$:

$$S_r^{(n+1)} = S_0^{\text{HCD}} + \Sigma_{0r}^{(n+1)}. \tag{22}$$

Since $S_0^{\text{HCD}} * S_0^{\text{HCD}} = 0$ and $\Sigma_{0r}^{(n+1)} * \Sigma_{0r}^{(n+1)} = 0$ by (21), then $S_r^{(n+1)}$ satisfies the WI (5) if

$$(S_0^{\text{HCD}} * \Sigma_{0r}^{(n+1)}) = 0. \tag{23}$$

From our knowledge of $\Sigma_{0r}^{(n+1)}$, (23) is just another way of writing the gauge invariance of S_0^{HCD}. Notice, as promised earlier, we have obtained this result without having to appeal to another regularisation scheme. The Ward identities with S_0 replaced by Σ_0 are strong enough to prove renormalisability, and hence to derive a renormalised and regulating action $S_r^{(n+1)}$ which itself satisfies the Ward identities.

3. Calculation of the one-loop β function

In this section we give the results of explicitly calculating the one-loop β function for the theory given by (1). Let us mention that to calculate the one-loop β function for supersymmetric theories, we could use RDR as the secondary regulator. Clearly, however, at this order in perturbation theory the results would not differ from those obtained using DR as the secondary regulator.

As mentioned in the introduction there is an absence from the literature of explicit calculations involving HCD in ordinary gauge theories and so one may regard the results presented here as not only a test of the combined HCD-DR system, but also of HCD themselves. Although the one-loop β function for supersymmetric theories in superspace regulated by HCD and PV has been carried out [3], the regularisation scheme here was 'implicit' in the sense that the β function was determined by D algebra and group theory alone, no momentum integrals being computed. To begin with we must determine the dependence of β on the renormalisation coefficients of a gauge theory which employs two regulators, in our case HCD and DR. As is usual we will use the $A_\mu{}^\alpha\bar\eta^\beta\eta^\gamma$ vertex to define the bare gauge coupling constant g_0:

$$g_0 = Z_1 g_r / Z_2 Z_3^{1/2} \tag{24}$$

where in (24) $Z_3^{1/2}$, $Z_2^{1/2}$ are the $A_\mu{}^\alpha$ and η^β wavefunction renormalisations, whilst $Z_1 g_r$ is the $A_\mu{}^\alpha \bar{\eta}_\beta \eta^\gamma$ vertex renormalisation, g_r being the renormalised coupling constant.

$\beta(g_r)$ is defined as

$$\beta(g_r) = \mu \frac{\delta g_r}{\delta \mu}\bigg|_{\text{fixed } \mathring{g}, \mathring{\alpha}, \mathring{\gamma}} \tag{25}$$

where $\hat{g}$ is dimensionless and μ an arbitrary renormalisation mass. Writing $g = \hat{g}_\mu{}^{2-\omega}$, we obtain for the HCD and DR scheme:

$$0 = (2-\omega)\frac{g_r Z_1}{Z_3^{1/2} Z_2} + \left(\beta \frac{\partial}{\partial \hat{g}_r} + \mu \frac{\partial \ln(\Lambda/\mu)}{\partial \mu} \frac{\partial}{\partial \ln(\Lambda/\mu)}\right) \frac{\hat{g}_r Z_1}{Z_3^{1/2} Z_2}. \tag{26}$$

We have derived this equation by taking $\mu\delta/\delta\mu$ of g_0 at fixed $\mathring{g}$, $\mathring{\alpha}$, $\mathring{\gamma}$, where $\circ$ refers to bare parameters.

Notice in (26) that, apart from the usual μ dependence of $\hat{g}_r$ from dimensional regularisation, there is also a dependence through $\ln(\Lambda/\mu)$ divergences in the Z coefficients. Equation (25) can be rewritten as

$$\beta(\hat{g}_r) = -(2-\omega)\frac{1}{(\partial/\partial \hat{g}_r)\ln(\hat{g}_r Z_1/Z_3^{1/2} Z_2)} + \frac{(\hat{g}_r Z_1/Z_3^{1/2} Z_2)_{,\ln(\Lambda/\mu)}}{(\hat{g}_r Z_1/Z_3^{1/2} Z_2)_{,g_r}}. \tag{27}$$

Now at one loop there are no mixed $(1/\varepsilon)\ln(\Lambda/\mu)$ divergences, so one can see that the only divergences contributing to the first term in (27) are those of $1/\varepsilon$ since the quantity $\varepsilon \ln(\Lambda/\mu)$ vanishes in the ordered limit $\varepsilon \to 0$; $\Lambda \to \infty$.

From now on we will take Green functions of the physical theory to be defined in this ordered limit. Therefore the first term in (27) is just the usual expression one obtains in dimensional regularisation. Similarly, only $\ln \Lambda/\mu$ divergences contribute to the second term in (27). If we expand the coefficients Z_1, Z_2, Z_3 as

$$\begin{aligned} Z_3 &= (1 + a_3/(2-\omega) + b_3 \ln(\Lambda/\mu)^2 + \cdots) \\ Z_2 &= (1 + a_2/(2-\omega) + b_2 \ln(\Lambda/\mu)^2 + \ldots) \\ Z_1 &= (1 + a_1/(2-\omega) + b_1 \ln(\Lambda/\mu)^2 + \ldots) \end{aligned} \tag{28}$$

then we arrive at an expression for β

$$\beta(\hat{g}_r) = (\hat{g}_r)^2(a_1' - \tfrac{1}{2}a_3' - a_2') + 2\hat{g}_r(b_1 - \tfrac{1}{2}b_3 - b_2). \tag{29}$$

In (29), $'$ denotes differentiation with respect to $\hat{g}_r$. We see that the β function of the two-regulator system is just the sum of β functions for each regulator. To compute it at one loop we have to determine the constants $a_1 \ldots b_3$ at that level. The graphs contributing to the a and b coefficients are listed in appendix 1, together with relevant momentum space Feynman rules. We find it convenient for later discussion to split each vertex into its HCD and non-HCD parts, the former being indicated by $\bigcirc$, the latter $\otimes$ at each vertex. One can see from the Feynman rules that the calculation of even one-loop diagrams is considerably complicated by the presence of HCD terms. These rules are calculated for a theory involving HCD in the gauge fields only, since as mentioned in § 2 naive power counting is not improved by their inclusion in other fields.

In computing the β function we will only be interested in those corrections to the non-HCD part of the gauge kinetic term which determine the value of Z_3; the diagrams in the appendix also give corrections to the HCD parts but these can be thought of as renormalisations of parameter γ (see (1)).

The results are set out in table 1, the details of which we will now discuss. Firstly the functions $A^{\tau}(k)$, $B^{\tau}(k)$ which appear in the coefficients a_3 and b_3 are quadratic in external momentum k_{μ}, where the label τ refers to the coefficient of the $k^2\delta_{\mu\nu}$ part.

Table 1. Overall factors of $(g/4\pi)^2$ are suppressed.

Graph	Divergences $\times 1/(2-\omega)$	Divergences $\times \ln(\Lambda/\mu)^2$	Group factors
I	0	$+\frac{3}{2}K^2$	$C_A\delta^{\alpha\beta}$
II			
(a)	0	0	
(b)	$B(k)$	$-B(k)$	$C_A\delta^{\alpha\beta}$
(c)	0	$+\frac{13}{24}k^2\delta_{\mu\nu}-\frac{16}{24}k_{\mu}k_{\nu}$	$C_A\delta^{\alpha\beta}$
(d)	0	0	
(e)	0	0	
(f)	$A(k)$	$-A(k)$	$C_A\delta^{\alpha\beta}$
(g)	$-\frac{4}{3}$	0	$C_F\delta^{ab}$
(h)	$+\frac{1}{24}K^2\delta_{\mu\nu}+\frac{1}{12}k_{\mu}k_{\nu}$	0	$C_A\delta^{\alpha\beta}$
III			
(a)	0	$+\frac{9}{4}k_{\mu}g$	$C_Af_{\alpha\beta\gamma}$
(b)	0	0	
(c)	0	$+\frac{1}{2}K_{\mu}g$	$C_Af_{\alpha\beta\gamma}$

We have not computed these because, as we shall now show, they do not affect the one-loop β function. The corrections A come from graph II(f) in appendix 1. This graph has two HCD vertex insertions and by power counting is divergent with respect to ε. In determining its $1/\varepsilon$ pole we find the result is proportional to

$$\left[\frac{(-1)^{2n}}{(2-n)!}\frac{1}{(2-\omega)}+\mathrm{O}(2-\omega)\right]\left[1+(\omega-2)\ln\left(\frac{\Lambda}{\mu}\right)^2+\mathrm{O}((\omega-2)^2)+\text{ finite terms}\right] \qquad (30)$$

for $n=1$ and 2.

From (30) we see that not only are there $1/\varepsilon$ divergences in II(c) but also a $\ln(\Lambda/\mu)^2$ has emerged from expanding about $\omega=2$. In non-HCD theories one obtains a $\ln(p^2/\mu^2)$ in this way which lead to on-shell infrared divergences. Here we find $\ln[(p^2+\Lambda^2)/\mu^2]$ instead, which for $\Lambda^2\to\infty$ yields a $\ln(\Lambda^2/\mu^2)$ divergence that has to be subtracted from the theory. The point of interest now is that looking at (30) we find that the residues of $1/(2-\omega)$ and $\ln(\Lambda/\mu)^2$ are related by a $-$ sign. Moreover (30) is the only source of $\ln(\Lambda/\mu)$ and $1/\varepsilon$ divergences in II(f). Therefore from the explicit expression for the β function, (26), we can see that contributions from the $\ln(\Lambda/\mu)^2$ and $1/(2-\omega)$ poles in II(f) cancel at the one-loop level. We find exactly the same phenomenon occurring for the graph II(b), where again calculation reveals a hidden $\ln(\Lambda/\mu)^2$ divergence that exactly cancels with the $1/(2-\omega)$ pole in the β function. We have labelled these corrections $B(k)$ in table 1.

The next observation we make from table 1 is that graphs II(d, e) and III(b), which we would expect to be Λ-divergent, are in fact finite. To understand why this should be the case we have to understand the origin of $\ln(\Lambda/\mu)$ divergences in diagrams that are regulated by HCD only. For these diagrams loop integration can be evaluated in

$d=4$, the result of which is of the generic form:

$$\int_0^1 dX_1 \ldots dX_K \frac{H(\Lambda, K, X_K)}{[\Lambda^2 g_1(X_K)+K^2 g_2(X_K)]^\alpha}\,\delta(1-X_1-\ldots-X_K). \tag{31}$$

In (31) X_K are a set of Feynman parameters, g_1, g_2 and H are functions, g_1, g_2 being linear, α is some integer and k_μ is the external momentum. (Note that we evaluate diagrams III at zero gluon momentum.) We know by power counting and gauge invariance that, if divergences occur in (31), they are logarithmic and hence given by those parts of H proportional to $\Lambda^{2\alpha}$. If we then look at leading-order terms in (31) in an expansion about $\Lambda=\infty$ we find it is given by a Λ-independent integral over some function $H^{2\alpha}$, say, of the Feynman parameters which is local in the momentum K. When Λ divergences occur they manifest themselves as singularities in this integral which gives terms like $\ln(X_K)|_0^1$. It is interesting to contrast this situation to dimensional regularisation in which ε poles appear in the expansion of $\Gamma(-n+\varepsilon)$, $n>0$, where the leading-order term $((-1)^n/n!)^n(1/\varepsilon)$ is dependent on ε.

The reason for these different behaviours is due to the dimensionality of Λ. However we can smuggle a Λ dependence into the poles $\ln X_K|_0^1$. This is achieved by smearing the limits 1, 0 with test functions $g(\Lambda, \mu)$, $h(\Lambda, \mu)$. We then demand that $g(\Lambda, \mu)$ be analytic in Λ as $\Lambda\to\infty$, and that its limit function be 1. Therefore, without loss of generality we can expand g in powers of the dimensionless quantity (μ/Λ), and similarly for h, whose limit function is taken to be 0. It is not difficult to prove that by doing this the $\ln X_K|_0^1$ poles are transferred to $\ln(\Lambda/\mu)$ poles as $\Lambda\to\infty$, and more importantly that the residue is *independent* of the coefficients in the Taylor expansion of the functions g and h. We may then feel satisfied that these manipulations render unique results.

Now for graphs II(d, e) and III(b) we can calculate the quantities g_1, g_2, $H^{2\alpha}$, and α; each time we find that the leading-order terms in (31) involve $\int_0^1 dX_K \ln f(X_K)$ where $f(X_K)$ is some linear functions of X_K that takes on the values of 1 or 0 at the limits. Such integrals are always regular. Hence for these graphs, the $\Lambda\to\infty$ limit is regular.

The net result of all this at the one-loop level at least is that the β function is completely determined by graphs which only include the original non-HCD vertices of the theory. The infinities in these graphs are extracted straightforwardly and they complete table 1. Let us mention at this point that graphs II contain Λ^2 divergences, which for separate graphs are non-zero. These are present in HCD regularisation due to the dimensional nature of Λ. However gauge invariance demands that they cancel in the complete one-loop correction.

The values of a_i, b_i are determined to be

$$a_1=0 \qquad a_2=0 \qquad a_3=(+\tfrac{1}{12}-2B^\tau-2A^\tau)C_A-\tfrac{4}{3}C_F$$
$$b_1=-\tfrac{11}{4}C_A \qquad b_2=-\tfrac{3}{2}C_A \qquad b_3=(+\tfrac{13}{12}+2B^\tau+2A^\tau)C_A. \tag{32}$$

In (32) we have suppressed overall factors of $(\hat{g}_r/4\pi)^2$. Substitution of (32) into (29) gives the standard result:

$$\beta(\hat{g}_r)=-\frac{(\hat{g}_r)^3}{16\pi^2}(\tfrac{11}{3}C_A-\tfrac{4}{3}C_F).$$

Although β functions calculated in different regularisation schemes need only be equivalent up to the addition of finite local counterterms, such insertions do not change the one-loop β function which should therefore be regularisation scheme independent. Our result is therefore consistent with this.

At this point we note that there is a consistency check on the mechanism which gave rise to cancellations of certain residues in the one-loop β function. By gauge invariance the Ward identities of the previous section imply that corrections to the non-HCD gauge inverse propagator should preserve the projection operator structure $K^2\delta_{\mu\nu} - K_\mu K_\nu$, for an arbitrary gauge choice α. For the two-regulator scheme this structure has to emerge in each of the two types of infinity. Therefore we have

$$(r_1 + A^\tau + B^\tau)\ln(\Lambda/\mu)^2 = -(r_2 + A^L + B^L)\ln(\Lambda/\mu)^2$$

and

$$(r_3 - A^\tau - B^\tau)\frac{1}{(2-\omega)} = -(r_4 - A^L - B^L)\frac{1}{(2-\omega)}. \tag{33}$$

In (33) the terms on the LHS are proportional to $k^2\delta_{\mu\nu}$ and those on the RHS to $k_\mu k_\nu$. A^L, B^L are just the $k_\mu k_\nu$ terms in the functions $A(k)$, $B(k)$. $r_1 \ldots r_4$ are corrections that come from ○ and mixed ○, ⊗ insertions, in the notation of appendix 1. Equation (33) implies the following constraint among the r:

$$(r_1 + r_3) = -(r_2 + r_4). \tag{34}$$

One can obtain the values of $r_1 \ldots r_4$ from table 1 and check that this constraint is indeed satisfied.

As we have previously mentioned supersymmetry requires the introduction of HCD for fermionic fields in the theory. It is therefore instructive to add such terms for the ψ and η fields to the action (1) and to calculate the one-loop β function as before. The results of calculating the divergent parts of two- and three-point functions are given in table 2. The quantities A, B, C, D, E, all quadratic functions of external momentum, have not been determined because as before their contributions cancel out of the β function at this order. With the results of table 2 we also conclude for this theory that the one-loop β function is completely determined by diagrams with the original (○) vertices in the theory, What is interesting here is that such diagrams are completely regulated by Λ, and so the effect of the secondary ε regulator drops out of the β function.

One can check that table 2 gives a_i, b_i coefficients that again give the standard result for $\beta(\hat{g}_r)$.

Once more we can use the gauge Ward identities to provide a consistency check on the 'cancellation mechanism' that occurs for each of the terms $A, B, \ldots, E$. From the transverse and longitudinal parts of the coefficients $A, \ldots, E$ we can derive a constraint as in (34). In this case $r_4 = r_3 = 0$, which implies that the sum of graphs II (a, c, h) preserve the projection operator structure of the inverse propagator. This is indeed verified by table 2.

It is natural to enquire whether this calculational result, that only (○) insertions into one-loop graphs affect the one-loop β function, generalises to higher orders. This turns out to be the case and a proof is given in appendix 2. The proof uses Weinberg's theorem [9] and also incorporates the cancellation mechanism between $1/(2-\omega)$ and $\ln(\Lambda/\mu)^2$ residues in the β function, which we observed in our one-loop calculations.

HCD, unlike what one might term 'passive' regularisation schemes such as dimensional regularisation, BPHZ and Pauli–Villars (which do not significantly alter

Table 2. Factors of $(g/4\pi)^2$ suppressed. We have compacted the notation in table 2 as follows. Factors of $\otimes^i$ following a graph mean i insertions of the vertex $\otimes$ into the graph in all possible ways.

Graph	Divergence $\times 1/(2-\omega)$	Divergence $\times \ln(\Lambda/\mu)^2$	Group factors
I			
(a)	0	$+\frac{1}{8}k^2$	$C_A\delta^{\alpha\beta}$
$\otimes^1$	0	0	
$\otimes^2$	$C(k)$	$-C(k)$	$C_A\delta^{\alpha\beta}$
II			
(a)	0	0	
(b)	$A(k)$	$-A(k)$	$C_A\delta^{\alpha\beta}$
(c)	0	$\frac{13}{24}k^2\delta_{\mu\nu}-\frac{16}{24}k_\nu k_\nu$	$C_A\delta^{\alpha\beta}$
(d)	0	0	
(e)	0	0	
(f)	$B(k)$	$-B(k)$	$C_A\delta^{\alpha\beta}$
(g)	0	$-\frac{4}{3}$	$C_F\delta^{ab}$
$\otimes^1$	0	0	
$\otimes^2$	$D(k)$	$-D(k)$	$C_F\delta^{ab}$
(h)	0	$\frac{1}{24}k^2\delta_{\mu\nu}+\frac{1}{12}k_\mu k_\nu$	$C_A\delta^{\alpha\beta}$
$\otimes^1$	0	0	
$\otimes^2$	$E(k)$	$-E(k)$	$C_A\delta^{\alpha\beta}$
III			
(a)	0	$+\frac{15}{8}k_\mu g$	$C_A\times f^{\alpha\beta\gamma}$
$\otimes^{1,2,3}$	0	0	
(c)	0	$-\frac{1}{2}k_\mu g$	$C_A\times f^{\alpha\beta\gamma}$
$\otimes^{1,2,3}$	0	0	

the original theory) change the interaction structure quite radically in gauge theories. We have shown however that the β function is determined by the interactions of the (unregularised) original theory. It is therefore reassuring to know that in calculating observable quantities like the β function HCD behave more like passive regulators.

We close this section by considering some consequences of our results. It is apparent from appendix 2 that some of our results of § 3 are not specific to the HCD-DR scheme but to HCD in general. In particular the result that graphs with $\otimes$ insertions and whose overall divergence is regularised by Λ do not correct β, remains true for the HCD-PV scheme. If one could show that the same was true of graphs with overall divergence regulated by PV (perhaps by a similar cancellation to the one occurring for HCD-DR, then our conclusions would hold in this regularisation scheme also. We have not verified that such cancellations occur for HCD-PV as the necessary calculations are rather complicated, so more work is required.

Interestingly our conclusions do not apply to a theory regulated by HCD and regularisation by dimensional reduction [11]. This is because the presence of ε-scalar loops within diagrams can yield factors of ε that can cancel with double poles of the type $(\ln(\Lambda/\mu))1/\varepsilon$ and hence contribute to the β function. No cancellation mechanism can occur here because we would require the presence of $1/\varepsilon$ poles which by our previous arguments are absent. Considering, however, the other difficulties surrounding RDR [12], we do not regard this situation as problematic.

From a practical point of view we have shown that β-function calculations in HCD regularisation schemes are considerably simpler than one would naively anticipate from the interactions of the regularised theory. This, together with their ability to preserve manifest supersymmetry, in our opinion makes such schemes worthy of further study.

Acknowledgments

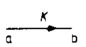
I would like to thank P West and P Howe for helpful discussions and motivation relating to this work and the SERC for financial support.

Appendix 1

The Euclidean momentum space Feynman rules are (for the choice $\gamma = 1$):

$$= -\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{K^2(1+\Lambda^{-4}K^4)}[\alpha = 1 \text{ gauge}].$$

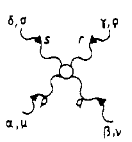
$$= -\frac{\delta_{\alpha\beta}}{K^2} \qquad \text{(A1.1)}$$



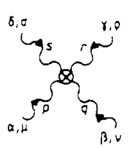
$$= \frac{\delta_{ab}}{K}$$

$$= -\frac{\mathrm{i}g}{6} f_{\alpha\beta\gamma}[(r-q)_\mu \delta_{\nu\rho} + (p-r)_\nu \delta_{\rho\mu} + (q-p)_\rho \delta_{\nu\mu}] \qquad \text{(A1.2)}$$

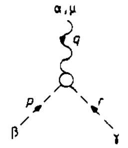
$$\begin{aligned} = \frac{-\mathrm{i}g}{\Lambda^4 4!}\{ & 4p^2(q+r)^2(p_\mu \delta_{\nu\rho}) + 4q^2(p+r)^2(q_\rho \delta_{\nu\mu}) \\ & + 4r^2(p+q)^2(r_\mu \delta_{\nu\rho}) \\ & + (2p\cdot r\delta_{\mu\rho} - p_{(\mu} r_{\rho)})[2p^2(q+2r)_\nu - 2r^2(q+2p)_\nu] \\ & - (2q\cdot r\delta_{\nu\rho} - q_\nu r_\rho)[2q^2(p+2r)_\mu - 2r^2(2q+p)_\mu] \\ & - (2q\cdot p\delta_{\mu\nu} - p_{(\mu} q_{\nu)})[2p^2(r+2q)_\rho - 2q^2(2p+r)_\rho]\} f_{\alpha\beta\gamma} \end{aligned} \qquad \text{(A1.3)}$$



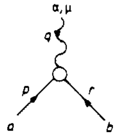
$$\begin{aligned} = g^2[& f_{\varepsilon\alpha\beta} f_{\varepsilon\gamma\delta} \delta_{\mu\rho} \delta_{\nu\sigma} \\ & - f_{\varepsilon\alpha\gamma} f_{\varepsilon\beta\delta} \delta_{\mu\sigma} \delta_{\rho\nu} \\ & + f_{\varepsilon\beta\gamma} f_{\varepsilon\alpha\delta} \delta_{\nu\mu} \delta_{\rho\sigma}] \end{aligned} \qquad \text{(A1.4)}$$



$$\begin{aligned} = \frac{1}{4} \times \frac{1}{4!} \frac{g^2}{\Lambda^4} [\![& f_{\varepsilon\alpha\beta} f_{\varepsilon\gamma\rho}\{(p+q)^2[(r+s)^2 \delta_{\mu\rho}\delta_{\nu\sigma} \\ & + 4s_\rho s_\mu \delta_{\nu\sigma}] + 4(r+s)^2(p+s)_\mu q_\rho \delta_{\nu\sigma} \\ & + (2q\cdot s\delta_{\nu\sigma} - q_{(\nu} s_{\sigma)})[(p+q)_\mu (r+3s)_\rho + q_\mu s_\rho] \\ & + 4p^2(r+s)_\nu p_\rho \delta_{\mu\sigma}\} \\ & + 2f_{\varepsilon\delta\alpha} f_{\varepsilon\beta\gamma} s^2(2r\cdot s\delta_{\mu\nu}\delta_{\rho\sigma} - r_{(\rho} s_{\sigma)} \delta_{\mu\nu} + 2(p+q+r)_\mu s_\nu \delta_{\rho\sigma})]\!]_{\mathrm{SYM}}. \end{aligned} \qquad \text{(A1.5)}$$



$$= \mathrm{i}gf_{\alpha\beta\gamma}(q+r)_\mu \tag{A1.6}$$

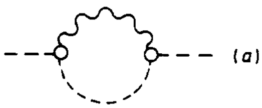


$$= \mathrm{i}g\rho^\mu (T^\alpha)_{ab} \qquad g = \hat{g}\mu^{2-\omega}. \tag{A1.7}$$

In (A1.5) ⟦ ⟧SYM means symmetrisation under the interchange of the triplets (α, μ, p); (β, ν, q); (γ, ρ, r); (δ, σ, s).

Diagrams contributing to the coefficients a_i, b_i; $i = 1 \ldots 3$ are

I



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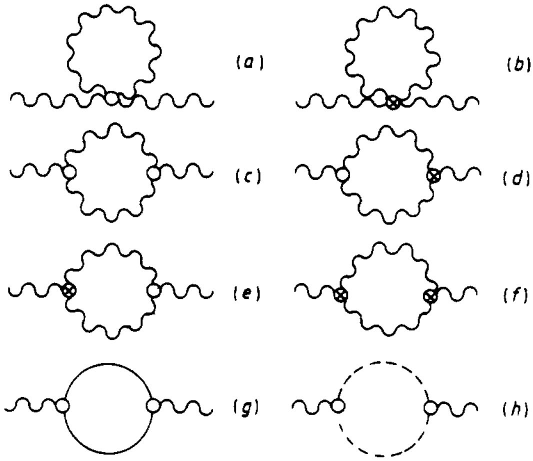
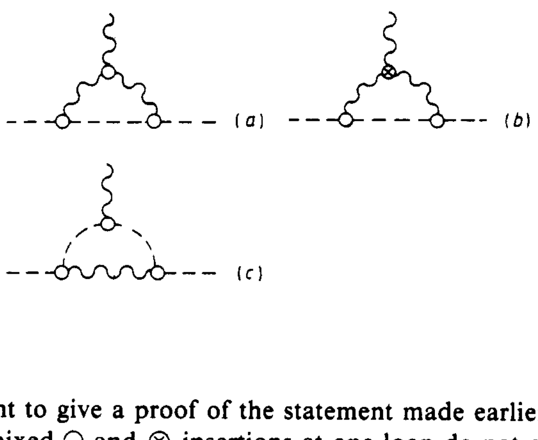


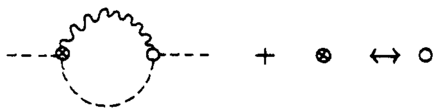
III



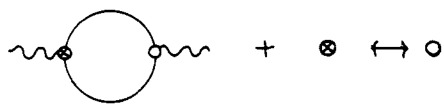
Appendix 2

Here we want to give a proof of the statement made earlier in the paper that graphs containing mixed ○ and ⊗ insertions at one loop do not correct the β function, the same being true for any number (>0) of ⊗ insertions for $L \geq 2$ loops.

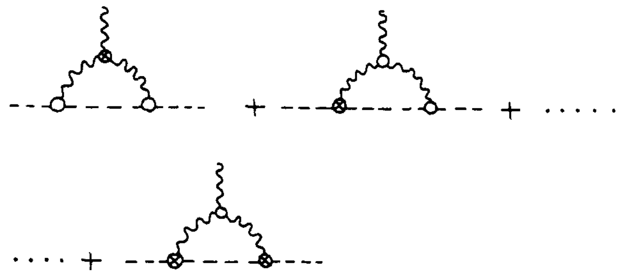
At one loop we will be interested in those graphs of appendix 1 with both ○ and ⊗ insertions along with



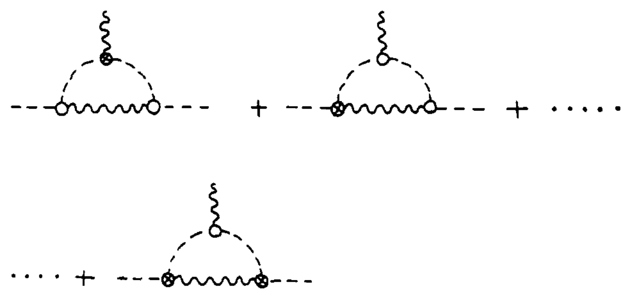
(A2.1)



(A2.2)



(A2.3)



(A2.4)

To be quite general we will take the degree HCD present in the fields A_μ, ψ and η to be P_i, $i = 1, 2, 3$. Up to now we have chosen $P_1 = 4$, $P_2 = P_3 = 0$. Propagators will typically have denominators of the form

$$[K^2(1 + \Lambda^{-P_i} K^{P_i})]^{-1} \qquad \text{(no sum on } i\text{)}.$$

which we can rewrite as

$$\frac{\Lambda^{P_i}}{K^2(\Lambda^{P_i} + K^{P_i})}.$$

Since we are interested in the $\Lambda \to \infty$ limit we keep track of the power of Λ in the numerators of the relevant graphs by introducing the quantity Q for each graph defined as

$$Q = \sum_i P_i I_i - P_i V_i \tag{A2.5}$$

where in (A2.5) I_i is the number of internal lines of species i, and V_i for $i = 1, 2, 3$ are the number of vertices in a graph originating from the new HCD terms we add to the action in the fields A_μ, ψ, η respectively. We take P_i to be even, so that we can factor the term $(\Lambda^{P_i}+K^{P_i})$ in the propagators as a product of terms like $(K^2\pm i\Lambda^2)$. One can therefore introduce $(\frac{1}{2}P_i+1)$ Feynman parameters $\{\chi_a\}$ for each such propagator occurring in a graph.

To begin with we will prove the one-loop statement. The result of computing a general one-loop momentum integral has the form

$$\int_0^1 dX_1\dots dX_{a-1}\frac{\Lambda^Q H(\{k\},\chi_a,\mu)}{[\Lambda^2 g_1(x_a)+g_2(\{k\},\chi_a,\mu)]^\alpha}. \tag{A2.6}$$

In (A2.6) $\{k\}$ represents a set of external momenta, μ a renormalisation mass, g_1, g_2 are some functions and a runs from 1 to N where N is given by

$$N=\sum_i(\tfrac{1}{2}P_i+1)I_i-1. \tag{A2.7}$$

By power counting, $\alpha\geqslant\frac{1}{2}Q$ with the divergent parts of (A2.6) coming from the equality. In the limit $\Lambda\to\infty$ we will only be interested in the X_a dependence of the functions H and g_1. At the one-loop level g_1 is a *linear* function of a subset $\{X_{a'}\}$ of the X_a where a' runs over $(I_i\frac{1}{2}P_i)$ independent values of X_a. The X_a dependence of H comes from the shifts in loop momentum carried out in computing the integral. It is therefore given by

$$L^n(\chi_a) \tag{A2.8}$$

where n is an integer ranging from 0 to the mass dimension of H, and $L(X_a)$ some linear function of X_a. The leading-order term as $\Lambda\to\infty$ of (A2.6) is

$$\int_0^1 dX_1\dots dX_{a-1}\frac{L^n(X_a)}{(g_1(X_{a'}))^{Q/2}}f_{\text{local}}(K). \tag{A2.9}$$

As mentioned previously, divergences in Λ manifest themselves as logarithmic terms that appear as a result of computing the integrals in (A2.9) and which diverge when evaluated at the limits. Such terms can only occur if the inverse power of the linear terms in $X_{a'}$ in (A2.9) is equal to a max, i.e. $\frac{1}{2}I_iP_i$. However (A2.9) gives this inverse power as $Q/2-n$. From (A2.5) we see that

$$Q/2-n=a'_{\max}-\sum_i P_iV_i-n. \tag{A2.10}$$

Hence logarithmic terms cannot be produced for $V_i\neq 0$, which is the case for the graphs under consideration. When $V_i=0$, logarithmic terms can appear. This corresponds to graphs having (○) vertex insertions only. Let us also mention that (A2.10) also holds for those graphs having ⊗ insertions only, if they are regulated by Λ. This implies that diagrams (A2.3) and (A2.4) with such insertions are also finite which agrees with our explicit calculations (see tables 1 and 2).

Now we consider the situation for $L\geqslant 2$ loops. Here we have to remember that the Lth-order, $(L-1)$th order counterterm insertions are present which change the naive power counting rules made from vertices in the regularised but unrenormalised theory. However it is easy to see that they are changed in as much as certain graphs with insertions of $\ln(\Lambda/\mu)$ counterterms are divergent with respect to ε. These graphs to the Lth loop order are just those ε-divergent one-loop diagrams with $(L-1)$th-order Λ counterterm insertions. There will also be diagrams with ε counterterm insertions. These are rendered ε finite as can be seen by considering power counting in the presence of ε counterterms. A similar phenomenon occurs for PV secondary regularisation [4]. One would naively think that by subtracting off the subdivergences of a graph by the inclusion of such counterterms that the remaining overall divergence is independent of the subdivergences, i.e. an exact cancellation has taken place between the counterterm and subdivergence. The worst that may happen if such a cancellation does not take place (even though the subdivergence may be removed) is the appearance of non-local so-called overlapping divergences. Caswell and Kennedy [8] have shown that this does not occur in the context of dimensional regularisation with minimal subtraction and, although it remains to be proved whether the same is true for HCD regularisation, we shall take it to be the case. Hidden in this last assumption are other subtleties that are required to prove the absence of overlapping divergences. The most important of these is that Weinberg's convergence theorem [9] holds for a renormalisation scheme involving minimal subtraction via counterterm insertions into the Lagrangian. In the original proof [10] Weinberg used the R operation of Bogoliubov and Parasiuk [10] to remove completely integrals associated with the subdivergence of a given graph, whereas in minimal subtraction one is *cancelling* only the pole part against a counterterm insertion [8].

The convergence theorem tells us that with the removal of all subdivergences, a graph Γ is absolutely convergent (in Euclidean space) if the degree of divergence $\delta(\Gamma)$ is <0, i.e. if its overall divergence is zero. To show that the β function is unchanged by graphs having an arbitrary (>0) number of $\otimes$ insertions we will therefore only concern ourselves with proving that their overall divergences vanish. We will take it that the counterterm graphs do their intended job and cancel all subdivergences without introducing any new spurious infinities, as is required by Weinberg's theorem.

Overall divergences occur when all the internal momentum of a graph become large and it is to these regions of momentum space that we will be interested in from now on. We want to know the overall divergence of a subtracted L-loop graph $\bar{R}(\Gamma_L)$, where $\bar{R}$ is defined as

$$\bar{R}(\Gamma_L)=\Gamma_L-T(\Gamma_L) \tag{A2.11}$$

where in (A2.11), Γ_L is the unrenormalised value of an L-loop graph and T is some subtraction operator (in our case minimal subtraction) that removes all subdivergences. In our case $T(\Gamma_L)$ consists of all the Lth order counterterm graphs to Γ_L. It will be useful in what follows to consider separating the set of graphs $\{\Gamma_L, T(\Gamma_L)\}$ into those whose overall divergence is regularised by Λ and those by ε. Consider first the former case and denote the set of graphs by $\{\Gamma_l^{(\Lambda)}\}$, l being the number of loops. Power counting and gauge invariance gives the Λ dependence of these graphs to be that of (A2.6). In this case the functions H, g_1 and g_2 will be non-linear in the Feynman parameters. We can imagine computing the l loops (for a given l) of $\Gamma_l^{(\Lambda)}$ one at a time and in doing so the X_a dependence of H is seen to come from translations of loop momenta. This allows us to factorise the X_a dependence of H as

$$\prod_l\left(\sum_{\nu=0}^{N_l}[U_l(X_{a_l})]^\nu\right). \tag{A2.12}$$

In (A2.12), U_l is a linear function of the Feynman parameters $\{X_{a_l}\}$. The number N_l measures the maximum number of loop momentum factors in the numerator of the lth-loop integral. The number of independent parameters $\{X_{a_l}\}$ entering the expression (A2.12) is bounded above by $\frac{1}{2}P_i(I_l)_i$ where $(I_l)_i$ is the generalisation of the previously defined I_i to the lth-loop subgraph.

We also find a factorisable form of g_1 at l loops

$$g_1(X)=\prod_l [g_l(X_{a'_l})]^{Ql/2} \tag{A2.13}$$

where in (A2.13) $g_l(X_{a'})$ are linear functions of the parameters $(X_{a'_l})$ entering the lth loop; $a'_l \leq \frac{1}{2}P_i I_{li}$ for all l. The number Q_l in (A2.13) is again the generalisation of (A2.5). The reason why we only require bounds on the parameters a'_l is because any logarithmic divergence that emerges from integration over all the Feynman parameters must appear when (A2.12) is lowest order in X_{a_l}, and when there are a maximum number of X_{a_l} in g_l of (A2.13). Consequently for the purposes of determining the overall behaviour of the l-loop graph as $\Lambda\to\infty$ we need only consider

$$\int \mathrm{d}X_1 \ldots \mathrm{d}X_{a'_l}[g_l(X_{a'_l})]^{-Ql/2}\tilde{f}_{\mathrm{local}}(K). \tag{A2.14}$$

In (A2.14) a'_l takes on its maximum value $\frac{1}{2}P_i I_{li}$. Logarithmic divergences may appear if

$$\tfrac{1}{2}Q_l=\tfrac{1}{2}P_i I_{li}. \tag{A2.15}$$

From (A2.5) we see that this equality is impossible to reach for $V_i \neq 0$. We can conclude that for any l-loop diagram $\Gamma_l^{(\Lambda)}$ having $\otimes$ insertions (so that $V_i\neq 0$) the limit $\Lambda\to\infty$ is regular when all loop momenta are hard. Again we see that the equality (A2.14) is reached for diagrams having (○) insertions only—which correspond to those of the original unregularised theory.

To complete our proof we have to consider those diagrams in the set $\{\Gamma_L, T(R_L)\}$ whose overall divergence is regulated by ε, denoted by $\{\Gamma_l^{(\varepsilon)}\}$. It is clear from our earlier discussion in this appendix that these diagrams only occur for $l=1$ and are just the one-loop ε-regulated graphs with $\ln(\Lambda/\mu)$ counterterm insertions. As such, their overall divergences do not vanish but nevertheless they do not alter the β-function because of the same cancellation mechanism that we saw appearing in the explicit calculations of § 2.

Hence we can conclude that for $L\geq 2$ loops, graphs having one or more $\otimes$ insertion do not contribute to the β function.

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